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Completed Beltrami–Michell Formulation for Analyzing Mixed Boundary Value Problems in Elasticity

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In elasticity, the method of forces, wherein stress parameters are considered as the primary unknowns, is known as the Beltrami–Michell formulation. The Beltrami–Michell formulation can only solve stress boundary value problems; it cannot handle the more prevalent displacement or mixed boundary value problems of elasticity. Therefore, this formulation, which has restricted application, could not become a true alternative to the Navier displacement method, which can solve all three types of boundary value problems. The restrictions of the Beltrami–Michell formulation have been alleviated by augmenting the classical formulation with a novel set of conditions identified as the boundary compatibility conditions. This new method, which completes the classical force formulation, has been termed the completed Beltrami–Michell formulation. The completed Beltrami–Michell formulation can solve general elasticity problems, with stress, displacement, and mixed boundary conditions in terms of stresses as the primary unknowns. The completed Beltrami–Michell formulation is derived from the stationary condition of the variational functional of the integrated force method. In the completed Beltrami–Michell formulation, stresses for kinematically stable structures can be obtained without any reference to displacements either in the field or on the boundary. This paper presents the completed Beltrami–Michell formulation and its derivation from the variational functional of the integrated force method. Examples are presented to demonstrate the applicability of the completed formulation for analyzing mixed boundary value problems under thermomechanical loads. Selected examples include analysis of a composite cylindrical shell, wherein membrane and bending response are coupled, and a composite circular plate.

Nomenclature

A	= strain energy in the integrated force method functional
a	= radius of shell
B	= complementary strain energy in the IFM functional
E	= modulus of elasticity
$[G]$	= material matrix
h	= plate or shell thickness
K	= plate or shell flexural rigidity
M_r, M_φ	= plate bending moments
M_x	= shell bending moment
N_φ	= shell tangential force
n_x, n_y	= directional cosines of the outward normal
P_x, P_y	= components of surface tractions
q	= intensity of the distributed load
$\mathcal{R}(\sigma)$	= boundary compatibility condition in terms of stresses
r, r_a, r_b	= radial coordinates
$\mathcal{S}_1(\sigma), \mathcal{S}_2(\sigma)$	= traction conditions in terms of stresses
t_0	= temperature at the midsurface
u, v, w	= displacement components
W	= potential of external loads in the IFM functional
x, y	= Cartesian coordinates
x_a, x_b	= coordinates of shell contours
α_t	= coefficient of thermal expansion
β	= cylindrical shell parameter
Γ	= boundary of an elastic continuum
Δt	= temperature difference between inner and outer surfaces
$\epsilon_x, \epsilon_y, \gamma_{xy}$	= strain tensor components for plane stress
ϵ_φ	= shell tangential strain
ν	= Poisson's ratio
Π_s^c	= IFM variational functional for cylindrical shells
Φ	= Airy's function for plane stress
Ψ	= shell stress function
$\Psi_p^{(q)}, \Psi_p^{(\Delta t)}$	= particular integral for mechanical and thermal loads, respectively
Ω	= shell domain

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Introduction

THE method of forces, also known as the Beltrami–Michell formulation (BMF), and its variant, the Airy's stress function formulation, were the preferred tools of analysis in elasticity during the 1940s and 1950s.[1,2] In fact, solutions for many classical elasticity problems were obtained using the method of forces.[1–3] The method of forces, however, could not compete with Navier's displacement formulation, especially in analyzing plates and shells with displacement and mixed boundary conditions. Thus, the application of the method of forces diminished and the displacement formulation gained popularity. The demise of the method of forces was not due to any intrinsic deficiency of the method but to the incompleteness of the formulation. Because a set of boundary equations was missing, the application of the classical BMF was restricted to solving only problems with stress boundary conditions. In other words, the Beltrami–Michell's force formulation can be applied to stress boundary value problems, but it cannot solve more prevalent displacement and mixed boundary value problems. The missing set of equations that completes the BMF has been identified as boundary compatibility conditions.[4] Augmentation of the classical BMF with the boundary compatibility conditions resulted in a novel force method, the completed Beltrami–Michell formulation (CBMF). The CBMF is as universal as the Navier's displacement formulation. It can solve all three classes of elasticity problems, stress, displacement, and mixed boundary value problems, thus overcoming the deficiency of the classical BMF. For kinematically stable structures, the CBMF can provide solution to stresses without any reference to the displacements, either in the field or on the boundary. In the CBMF, stresses are obtained directly as a solution to a set of equations. Displacements, if required, can be obtained by integrating stresses. Thus, the CBMF can provide accurate solutions for both stresses and displacements. In the Navier's method, displacements (whether required or not) must be calculated first; then stresses are determined through differentiation. As a result, in the displacement method stresses can become inaccurate when approximate techniques are used. In the CBMF, thermal and initial strains are directly handled through the compatibility formulation. In the displacement method these have to be treated indirectly using the concept of equivalent loads.

Justifications to develop the CBMF include the following: 1) The method of forces can compete or become an alternative to the displacement method, provided its limitations are eliminated. The CBMF which can solve all three classes of boundary value problems alleviates a limitation of the classical formulation. 2) The accuracy of stress solutions can be improved provided all of the equations of structural mechanics are known, understood, and utilized. The missing boundary compatibility conditions, therefore, need to be investigated and utilized without qualification. 3) All solutions that have been obtained using the classical BMF need to be verified; that is, it must be determined whether the boundary compatibility conditions have been satisfied or not. Noncompliance of boundary compatibility conditions for a problem is indicated in Ref. 4. 4) The integrated force method (IFM), which is a discretized version of the CBMF, provides accurate stress solutions even for coarse finite element meshes.[5,6] The IFM exhibits potential to eliminate several shortcomings of the displacement method, such as indirect stress calculations, circuitous treatment of initial deformations, shear locking, and stress inaccuracy in stress concentration zones; to simplify design sensitivity analysis[7,8] and alleviate singularities in structural optimization[9]; etc. The CBMF provides the theoretical background for the IFM formulations.

The boundary compatibility conditions have been reported for plane stress problems,[5] and analyses using boundary compatibility conditions were published for rectangular[4] and circular plates[10] in flexure for mechanical loads. This paper includes the formal presentation of the CBMF and its application to composite circular plates and circular cylindrical shells under thermomechanical loads. The CBMF for plates and shells is derived from the IFM variational functional. Mixed boundary value problems are solved to demonstrate the capability of the formulation. A composite plate subjected to thermomechanical loads is analyzed to demonstrate the application of the CBMF to problems with displacement and interface (or jump) boundary conditions. A composite cylindrical shell subjected to thermomechanical loads is analyzed next. The shell example demonstrates the applicability of the CBMF when membrane and bending responses are coupled. In addition, this paper is to serve as an initial, yet an unified and systematic, attempt to bring back the method of forces for analyzing general elastic continua.

Completed Beltrami–Michell Formulation of Elasticity

Basic concepts of the method of forces (the CBMF being its specialization for elasticity) can be initiated from the stress strain law which is universal to all analysis formulations. Hooke's law that links stresses $\{\sigma\}$ to strains $\{\epsilon\}$ through a material matrix $[G]$ can be written as

$$\{\sigma\} = [G]\{\epsilon\} \tag{1}$$

The stresses in Eq. (1) must satisfy the state of equilibrium, and the strains must satisfy compatibility conditions. For an elastic continuum, stresses and strains must satisfy equilibrium equations and compatibility conditions, respectively, both in the field and on the boundary. These equations can be divided into four groups: Ia) stress equilibrium equations in the field, Ib) stress equilibrium equations on the boundary (or traction conditions), IIa) strain compatibility conditions in the field, and IIb) strain compatibility conditions on the boundary. The equation set of the classical BMF contained equations Ia, Ib, and IIa but it missed the boundary compatibility conditions. The CBMF utilizes all four groups of equations Ia, Ib, IIa, and IIb.

Governing Equations for the Completed Beltrami–Michell Formulation

Basic concepts of the CBMF are presented for a plane stress problem. For simplicity and clarity, homogeneous kinematic boundary conditions are considered and initial deformations, along with body forces, are neglected. The equations of equilibrium in the field (group Ia) are $\partial\sigma_x/\partial x + \partial\tau_{xy}/\partial y = 0$ and $\partial\tau_{xy}/\partial x + \partial\sigma_y/\partial y = 0$; and the equilibrium conditions on the boundary or traction conditions (group Ib) are $\mathcal{S}_1(\sigma) = \sigma_x n_x + \tau_{xy} n_y - P_x = 0$, and $\mathcal{S}_2(\sigma) = \tau_{xy} n_x + \sigma_y n_y - P_y = 0$. Here σ_x, σ_y, and τ_{xy} are three components of the stress tensor; n_x and n_y are the direction cosines of the outward normal vector; and P_x and P_y are prescribed boundary tractions. The equilibrium equations are functionally indeterminate[9] both in the field and on the boundary, because three unknown stresses are expressed in terms of two equations (groups Ia and Ib), respectively. The functional indeterminacy of stresses in the domain is alleviated through the field compatibility condition of St. Venant (group IIa), which can be written in terms of stress components as $\nabla^2(\sigma_x + \sigma_y) = 0$.

The functional indeterminacy on the boundary, which made the BMF incomplete, has been alleviated with the formulation of the boundary compatibility condition. This boundary condition, when expressed in terms of stresses, has the following form for plane stress problem:

$$\mathcal{R}(\sigma) = \frac{\partial(\sigma_y - \nu\sigma_x)}{\partial x} n_x + \frac{\partial(\sigma_x - \nu\sigma_y)}{\partial y} n_y - (1+\nu)\left(\frac{\partial\tau_{xy}}{\partial x} n_y + \frac{\partial\tau_{xy}}{\partial y} n_x\right) = 0 \tag{2}$$

The set of three equations consisting of two traction conditions (group Ib) and one boundary compatibility condition given in Eq. (2) ensures functional determinacy of stresses on the boundary because three unknown stresses are expressed in terms of three equations. Equations of equilibrium (groups Ia and Ib) and the field compatibility condition (group IIa) together with the boundary compatibility condition given in Eq. (2) ensure the functional determinacy of stresses both in the field and on the boundary of an elastic continuum and represent the CBMF. The CBMF can solve a general problem with stress, displacement, or mixed boundary conditions.

The stationary condition of the IFM variational functional also yields two displacement boundary conditions given as $u = \bar{u} = 0$, and $v = \bar{v} = 0$. The displacement calculations from stresses utilize displacement boundary conditions to evaluate constants of integration.[4,10,11]

Completed Beltrami–Michell Formulation Solution Strategy for Composite Continuum

The CBMF solution steps for a composite elastic continuum with fields Ω_1 and Ω_2 and boundaries Γ_s, Γ_u, and Γ_t shown in Fig. 1 are briefly described.

1) Satisfy field equilibrium and field compatibility conditions (groups Ia and IIa) for both domains Ω_1 and Ω_2. (In the displacement formulation the Navier's field equations[3] have to be satisfied.)

2) Satisfy traction boundary conditions (group Ib) and boundary compatibility condition given in Eq. (2) on contours Γ_s and Γ_u, respectively. (In the displacement method equivalent traction conditions and displacement boundary conditions on contours Γ_s and Γ_u, respectively, have to be satisfied.)

3) On the interface boundary Γ_t three conditions have to be satisfied: two residual equilibrium equations,

$$\mathcal{S}_1^I(\sigma) - \mathcal{S}_1^{II}(\sigma) = 0 \tag{3a}$$

$$\mathcal{S}_2^I(\sigma) - \mathcal{S}_2^{II}(\sigma) = 0 \tag{3b}$$

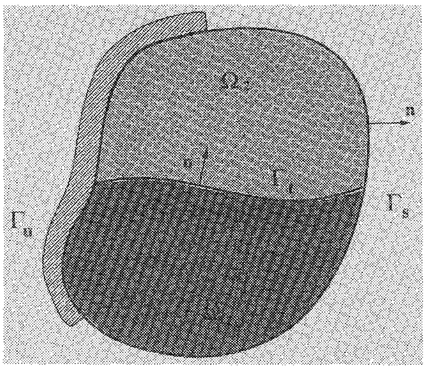


Fig. 1 Composite elastic continuum.

and one residual compatibility condition,

$$\mathcal{R}^I(\sigma) - \mathcal{R}^{II}(\sigma) = 0 \tag{4}$$

where the superscripts I and II denote the domains, Ω_1 and Ω_2, respectively. (In Navier's method displacement and traction conditions have to be satisfied at the interface.)

4) Once the solution for stresses has been obtained, displacements, if required, can be calculated by integration. The evaluation of integration constants requires the kinematic boundary conditions.

The composite structure can be solved by the CBMF or by the Navier's displacement method. The problem, however, cannot be solved by the classical BMF because of lack of boundary compatibility conditions for boundary Γ_u and the interface contour Γ_t.

Properties of Compatibility Conditions

Two important properties of compatibility conditions for a plane stress problem are as follows.

1) The field compatibility condition, when expressed in terms of displacements u and v becomes a trivial constraint, such as an identity $[f(u, v) - f(u, v)] = 0$, where f represents the field compatibility condition. The boundary compatibility condition given in Eq. (2), however, does not become a trivial equation when written in terms of displacements. In terms of displacements the boundary compatibility condition becomes

$$\left[\frac{\partial^2 v}{\partial x \partial y} - \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right)\right] n_x + \left[\frac{\partial^2 u}{\partial x \partial y} - \frac{1}{2}\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y}\right)\right] n_y = 0 \tag{5}$$

The nontrivial property of the boundary compatibility condition contradicts a popular belief that all compatibility conditions are automatically satisfied in the displacement method.

2) The field compatibility condition can be derived from the strain displacement relations. This logic as yet cannot be extended to derive the boundary compatibility conditions. The boundary compatibility conditions can be generated only from the IFM variational functional.

Applications of the Completed Beltrami–Michell Formulation

Application of the CBMF for thermomechanical stress analyses is illustrated for composite plates and composite cylindrical shells with mixed boundary conditions.

Completed Beltrami–Michell Formulation Equations for Bending of Circular Plates

The IFM variational functional[4] for circular plates is extended to incorporate the thermal effects. The stationary condition of this modified functional yields the CBMF equations as follows.

Field equation of equilibrium:

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rM_r) - \frac{\mathrm{d}M_\varphi}{\mathrm{d}r} + rq = 0 \tag{6}$$

Field compatibility condition:

$$r\frac{\mathrm{d}}{\mathrm{d}r}(M_\varphi - \nu M_r) + (1+\nu)(M_\varphi - M_r) + Kr\frac{\alpha_t}{h}\frac{\mathrm{d}\Delta t}{\mathrm{d}r} = 0 \tag{7}$$

where M_r and M_φ are the radial and tangential moment, respectively; r is the radial coordinate; q is the intensity of the distributed load; h is the plate thickness; material constant K is defined as $K = Eh^3/12(1-\nu^2)$; E is the modulus of elasticity of the material; ν is Poisson's ratio; α_t is the thermal coefficient of the material; and Δt is the temperature difference between the upper and the lower surface of the plate.

Boundary conditions are specialized for various support conditions: simply supported contour

$$M_r = 0 \tag{8}$$

and clamped contour

$$(1/K)(M_\varphi - \nu M_r) + \alpha_t \Delta t / h = 0 \tag{9}$$

Equation (9) represents a boundary compatibility condition. The interface conditions for a composite domain have the following form:

$$M_r^I = M_r^{II} \tag{10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(rM_r^I\right) - M_\varphi^I = \frac{\mathrm{d}}{\mathrm{d}r}\left(rM_r^{II}\right) - M_\varphi^{II} \tag{10b}$$

$$\frac{1}{K^I}\left(M_\varphi^I - \nu^I M_r^I\right) + \alpha_t^I \frac{\Delta t}{h^I} = \frac{1}{K^{II}}\left(M_\varphi^{II} - \nu^{II} M_r^{II}\right) + \alpha_t^{II}\frac{\Delta t}{h^{II}} \tag{10c}$$

Equation (10c) represents the interface compatibility condition.

Analysis of a Composite Circular Plate under Thermomechanical Loads

The CBMF solution procedure is presented through the analysis of a composite plate shown in Fig. 2. The plate consists of two segments: an inner plate, Ω_i, with radius a, material properties E_i, ν_i, and thickness h_i; and an outer annular plate, Ω_o, with inner radius a, outer radius b, material properties E_o, ν_o, and thickness h_o. The inner plate is subjected to a uniformly distributed mechanical load of intensity q, and the outer plate is exposed to an uneven heating with the temperature difference Δt. The plate is clamped at the outer contour, $r = b$. This example illustrates the CBMF solution process for 1) using the boundary compatibility condition at a clamped contour, 2) analyzing composite domains using transition conditions, and 3) analyzing thermomechanical loads.

Equations (6) and (7) are solved to obtain general expressions for the moments M_r and M_φ for the regions Ω_i and Ω_o, respectively, as

$$M_r^i(r) = -\left(B_1/r^2\right) + \tfrac{1}{2}C_1(1+\nu_i)\log r + \tfrac{1}{4}C_1(1-\nu_i) + \tfrac{1}{2}D_1 - \tfrac{1}{16}(3+\nu_i)qr^2 \tag{11a}$$

$$M_\varphi^i(r) = \left(B_1/r^2\right) + \tfrac{1}{2}C_1(1+\nu_i)\log r - \tfrac{1}{4}C_1(1-\nu_i) + \tfrac{1}{2}D_1 - \tfrac{1}{16}(1+3\nu_i)qr^2 \tag{11b}$$

$$M_r^o(r) = -\left(B_2/r^2\right) + \tfrac{1}{2}C_2(1+\nu_o)\log r + \tfrac{1}{4}C_2(1-\nu_o) + \tfrac{1}{2}D_2 \tag{11c}$$

$$M_\varphi^o(r) = \left(B_2/r^2\right) + \tfrac{1}{2}C_2(1+\nu_o)\log r - \tfrac{1}{4}C_2(1-\nu_o) + \tfrac{1}{2}D_2 \tag{11d}$$

where B_1, C_1, D_1 and B_2, C_2, D_2 are six integration constants. These six constants are calculated from the following six conditions: one boundary compatibility condition given in Eq. (9) at the clamped outer contour ($r = b$), three transition conditions given in Eqs. (10) at the interface ($r = a$); and two implicit conditions (M_r and M_φ are

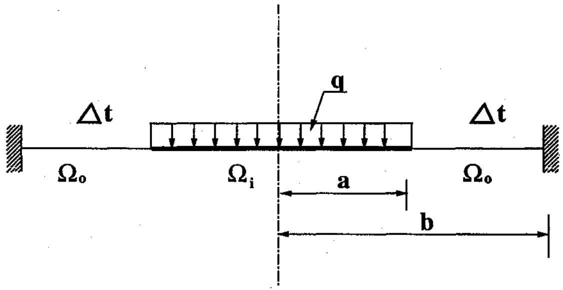


Fig. 2 Composite circular plate subjected to a uniform load q and temperature Δt.

finite) at the origin ($r = 0$). The solution is obtained for numerical values for the material parameters $E_i = 10.6 \times 10^6$ psi, $\nu_i = 0.33$, and $\alpha_t^{(i)} = 12.6 \times 10^{-6} 1/°\text{F}$, and $E_o = 30.0 \times 10^6$ psi, $\nu_o = 0.30$, and $\alpha_t^{(o)} = 6.3 \times 10^{-6} 1/°\text{F}$. The radii are $a = 6$ in. and $b = 12$ in., the thicknesses are $h_i = 0.2$ in. and $h_o = 0.15$ in., the magnitude of the distributed load is $q = 100$ lb/in.^{2}, and the temperature difference is $\Delta t = 50°$F. The final solution for M_r and M_φ is for the domain Ω_i $(0 \leq r \leq a)$

$$M_r^i(r) = 844.05 - 20.81r^2 \tag{12a}$$

$$M_\varphi^i(r) = 844.05 - 12.44r^2 \tag{12b}$$

and for the domain Ω_o $(a \leq r \leq b)$

$$M_r^o(r) = 2046.63 - (5203.06/r^2) - 1170 \log r \tag{12c}$$

$$M_\varphi^o(r) = 2676.63 + (5203.06/r^2) - 1170 \log r \tag{12d}$$

The displacements are obtained by integrating the moment-curvature relations. The displacement continuity conditions are used to evaluate constants of integration. Displacements for the domains Ω_i and Ω_o, are given as

$$w^i(r) = 3.1209 - 0.0356r^2 + 0.1757 \times 10^{-3} r^4 \tag{13a}$$

$$w^o(r) = 5.3614 - 0.1344r^2 - 0.7296 \log r + 0.04417r^2 \log r \tag{13b}$$

respectively. The solutions for the moments and displacement were verified through the displacement method.

Integrated Force Method Variational Formulation for Cylindrical Shells

The CBMF for a cylindrical shell is obtained from the stationary condition of the IFM functional Π_s^c defined as

$$\Pi_s^c = A + B - W \tag{14}$$

where the strain energy A, the complementary energy B, and the work of external forces W for thermomechanical loads are given as

$$A = \int_\Omega \left[M_x \left(-\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + (1+\nu)\alpha_t \frac{\Delta t}{h} \right) + N_\varphi \left(-\frac{w}{a} + \alpha_t t_0 \right) \right] \mathrm{d}\Omega \tag{15a}$$

$$B = \int_\Omega \left[\Psi \left(\frac{M_x}{K} + (1+\nu)\alpha_t \frac{\Delta t}{h} \right) - a \frac{\mathrm{d}^2 \Psi}{\mathrm{d}x^2} \left(\frac{N_\varphi}{Eh} + \alpha_t t_0 \right) \right] \mathrm{d}\Omega \tag{15b}$$

$$W = \int_\Omega q w \, \mathrm{d}\Omega \tag{15c}$$

where w is the radial displacement, M_x the bending moment, N_φ the tangential force, q the distributed load, h the thickness of the shell, $K = Eh^3/12(1-\nu^2)$ the rigidity, t_0 the temperature at the midsurface of the shell, and Ψ the stress function. The stress function is defined through a Washizu's procedure[12] as

$$M_x = \Psi \tag{16a}$$

$$N_\varphi = -a \left(\frac{\mathrm{d}^2 \Psi}{\mathrm{d}x^2} + q \right) \tag{16b}$$

The variation $\delta \Pi_s^c$ has the following form:

$$\begin{aligned} \delta \Pi_s^c = & -\int_\Omega \left[\frac{\mathrm{d}^2 M_x}{\mathrm{d}x^2} + \frac{1}{a} N_\varphi + q \right] \mathrm{d}\Omega \delta w \\ & - \int_\Omega \left[\frac{M_x}{K} - \frac{a}{Eh} \frac{\mathrm{d}^2 N_\varphi}{\mathrm{d}x^2} + (1+\nu)\alpha_t \frac{\Delta t}{h} - a\alpha_t \frac{\mathrm{d}^2 t_0}{\mathrm{d}x^2} \right] \mathrm{d}\Omega \delta \Psi \\ & + 2a\pi \left[M_x \delta \left(-\frac{\mathrm{d}w}{\mathrm{d}x} \right) + \frac{\mathrm{d}M_x}{\mathrm{d}x} \delta w \right]_{x_a}^{x_b} \\ & + 2a\pi \left[a \left(\frac{N_\varphi}{Eh} + \alpha_t t_0 \right) \delta \frac{\mathrm{d}\Psi}{\mathrm{d}x} - a \left(\frac{1}{Eh} \frac{\mathrm{d}N_\varphi}{\mathrm{d}x} + \alpha_t \frac{\mathrm{d}t_0}{\mathrm{d}x} \right) \delta \Psi \right]_{x_a}^{x_b} \end{aligned} \tag{17}$$

The stationary condition of the variational functional with respect to the displacement w and the stress function Ψ yields all of the equations of the CBMF as follows.

Field equation of equilibrium:

$$\frac{1}{a} N_\varphi + \frac{\mathrm{d}^2 M_x}{\mathrm{d}x^2} + q = 0 \tag{18}$$

Field compatibility condition:

$$\frac{M_x}{K} - \frac{a}{Eh} \frac{\mathrm{d}^2 N_\varphi}{\mathrm{d}x^2} + (1+\nu)\alpha_t \frac{\Delta t}{h} - a\alpha_t \frac{\mathrm{d}^2 t_0}{\mathrm{d}x^2} = 0 \tag{19}$$

Contour terms in Eq. (17) yield boundary conditions as follows.

Free contour: Moment and its derivative vanish because $\delta w \neq 0$ and $\delta(\mathrm{d}w/\mathrm{d}x) \neq 0$

$$M_x = 0 \tag{20a}$$

$$\frac{\mathrm{d}M_x}{\mathrm{d}x} = 0 \tag{20b}$$

Simply supported contour: The rotation is allowed on a simply supported boundary; hence,

$$M_x = 0 \tag{21a}$$

The variation of the derivative of the stress function, $\delta(\mathrm{d}\Psi/\mathrm{d}x)$ is not zero; hence,

$$N_\varphi / Eh + \alpha_t t_0 = 0 \tag{21b}$$

Equation (21b) represents a boundary compatibility condition for a simply supported boundary.

Clamped contour: For the clamped contour two compatibility conditions, which are coefficients of $\delta\Psi$ and $\delta(\mathrm{d}\Psi/\mathrm{d}x)$ in the contour terms in Eq. (17), must be satisfied:

$$\frac{N_\varphi}{Eh} + \alpha_t t_0 = 0 \tag{22a}$$

$$\frac{1}{Eh} \frac{\mathrm{d}N_\varphi}{\mathrm{d}x} + \alpha_t \frac{\mathrm{d}t_0}{\mathrm{d}x} = 0 \tag{22b}$$

Transition conditions: A composite shell has two residual equilibrium conditions,

$$M_x^{(I)} - M_x^{(II)} = 0 \tag{23a}$$

$$\frac{\mathrm{d}M_x^{(I)}}{\mathrm{d}x} + \frac{\mathrm{d}M_x^{(II)}}{\mathrm{d}x} = 0 \tag{23b}$$

and two residual boundary compatibility conditions,

$$\left(\frac{N_\varphi^{(I)}}{E_1 h_1} + \alpha_t^I t_0\right) - \left(\frac{N_\varphi^{(II)}}{E_2 h_2} + \alpha_t^{II} t_0\right) = 0 \quad (23c)$$

$$\left(\frac{1}{E_1 h_1}\frac{\mathrm{d}N_\varphi^{(I)}}{\mathrm{d}x} + \alpha_t^I \frac{\mathrm{d}t_0}{\mathrm{d}x}\right) + \left(\frac{1}{E_2 h_2}\frac{\mathrm{d}N_\varphi^{(II)}}{\mathrm{d}x} + \alpha_t^{II}\frac{\mathrm{d}t_0}{\mathrm{d}x}\right) = 0 \quad (23d)$$

The field equations (18) and (19), together with appropriate boundary conditions, represent the sufficient number of equations for the solution of stress variables. The boundary compatibility conditions given in Eqs. (22) are derived for the first time. Without the boundary compatibility conditions the solution for the shell in terms of the stress variables cannot be obtained for displacement or for mixed boundary conditions. The transition conditions given in Eqs. (23c) and (23d) enable the solution of composite shells by the CBMF.

The field equations (18) and (19) may be uncoupled to obtain

$$\frac{\mathrm{d}^4 M_x}{\mathrm{d}x^4} + 4\beta^4 M_x = -\frac{\mathrm{d}^2 q}{\mathrm{d}x^2} - \frac{Eh}{a^2}\alpha_t\left[(1+\nu)\frac{\Delta t}{h} + a\frac{\mathrm{d}^2 t_0}{\mathrm{d}x^2}\right] \quad (24a)$$

$$N_\varphi = -a\left(\frac{\mathrm{d}^2 M_x}{\mathrm{d}x^2} + q\right) \quad (24b)$$

or

$$\frac{\mathrm{d}^4 N_\varphi}{\mathrm{d}x^4} + 4\beta^4 N_\varphi = -\frac{Eh}{Ka}q + \frac{Eh}{a}\alpha_t\left[(1+\nu)\frac{1}{h}\frac{\mathrm{d}^2\Delta t}{\mathrm{d}x^2} - a\frac{\mathrm{d}^4 t_0}{\mathrm{d}x^4}\right] \quad (25a)$$

$$M_x = \frac{Ka}{Eh}\frac{\mathrm{d}^2 N_\varphi}{\mathrm{d}x^2} + K\alpha_t\left[(1+\nu)\frac{\Delta t}{h} - a\frac{\mathrm{d}^2 t_0}{\mathrm{d}x^2}\right] \quad (25b)$$

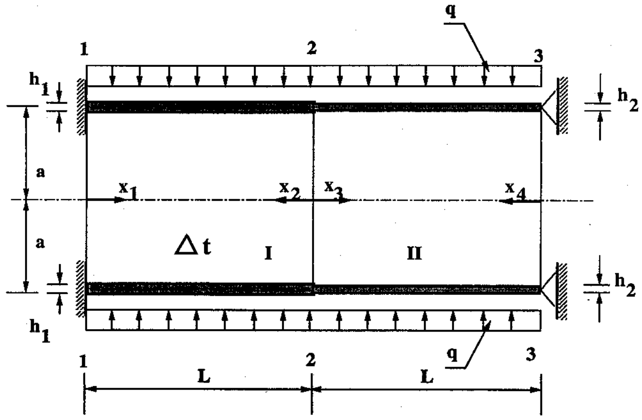
where $\beta^4 = 3(1-\nu^2)/a^2h^2$. Either Eq. (24a) or (25a) can be used for solution. Here, the moment equation Eq. (24) is selected. Its general solution is

$$M_x = C_1 \cosh\beta x + C_2 \sinh\beta x + C_3 \cos\beta x + C_4 \sin\beta x + \Psi_p^{(q)} + \Psi_p^{(\Delta t)} \quad (26)$$

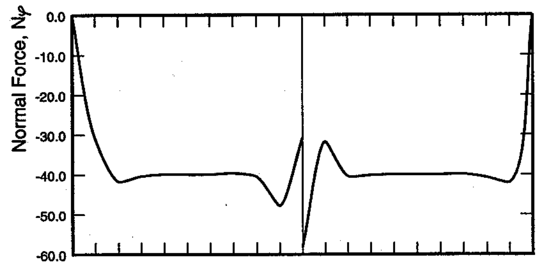
where C_1, C_2, C_3, and C_4 are the constants of integration, and $\Psi_p^{(q)}$ and $\Psi_p^{(\Delta t)}$ are particular integrals for distributed loads and temperature, respectively. Once M_x is known, N_φ can be calculated by back substitution from Eq. (24b).

Analysis of a Long Composite Shell

A composite cylindrical shell of radius a, and length $2L$, made of two different materials, is shown in Fig. 3a. Region I, bounded by contours 1–1 and 2–2, has material parameters E_1 and ν_1, and thickness h_1; and region II, bounded by contours 2–2 and 3–3, has material and geometric properties E_2, ν_2, and h_2. The shell is clamped along the contour 1–1, and simply supported along the contour 3–3. Both regions are subjected to a uniformly distributed load of intensity q. Region I is also subjected to a temperature change of Δt. The solution for a composite shell is obtained by superposing the two component solutions. Each component solution involves four integration constants; hence, there are a total of eight unknowns for the composite shell. The eight constants of integration are evaluated from the following eight conditions: two boundary compatibility conditions for the clamped boundary 1–1 given in Eqs. (22a) and (22b), two boundary conditions for the simple supported boundary 3–3 given in Eqs. (21a) and (21b), and four transition conditions at the interface 2–2 given in Eqs. (23a–23d). For simplicity it is assumed that the material and geometric parameters for both regions are such that the products $\beta_1 L \geq 5$ and $\beta_2 L \geq 5$, which allows both regions to be treated as long shells. Consequently, the response for the composite shell can be obtained by superposing effects from the three boundaries, that is, from the simply supported boundary, clamped boundary, and the interface boundary, as shown in Fig. 3a.



a) Composite cylindrical shell



b) Normal force distribution along the x axis

Fig. 3 Analysis of a composite cylindrical shell.

Response near Clamped Boundary: Contour 1–1

The expressions for the moment M_x and the force N_φ are first derived for the edge effects on contour 1–1. The coordinate system is defined such that the axis x_1 is placed along the axis of the shell, with the origin in the plane defined by contour 1–1. Using the boundary conditions given in Eqs. (21a) and (21b) the solution for M_x and N_φ is obtained as

$$M_x(x_1) = \left(q/2\beta_1^2\right)e^{-\beta_1 x_1}(\sin\beta_1 x_1 - \cos\beta_1 x_1) + \Psi_p^{(\Delta t)} \quad (27a)$$

$$N_\varphi(x_1) = aq\left[e^{-\beta_1 x_1}(\cos\beta_1 x_1 + \sin\beta_1 x_1) - 1\right] \quad (27b)$$

Using Eq. (27b), the expression for the displacement w is obtained as

$$w(x_1) = -\left(qa^2/Eh_1\right)\left[e^{-\beta_1 x_1}(\cos\beta_1 x_1 + \sin\beta_1 x_1) - 1\right] \quad (28)$$

Response near Interface: Contour 2–2

The expressions for the moment M_x and the force N_φ defined for regions I and II, respectively, are obtained as

$$M_x^{(I)}(x_2) = e^{-\beta_1 x_2}(A_1 \cos\beta_1 x_2 + B_1 \sin\beta_1 x_2) + \Psi_p^{(\Delta t)} \quad (29a)$$

$$N_\varphi^{(I)}(x_2) = -a\left[2\beta_1^2 e^{-\beta_1 x_2}(-B_1\cos\beta_1 x_2 + A_1 \sin\beta_1 x_2) + q\right] \quad (29b)$$

$$M_x^{(II)}(x_3) = e^{-\beta_2 x_3}(A_2\cos\beta_2 x_3 + B_2 \sin\beta_2 x_3) \quad (29c)$$

$$N_\varphi^{(II)}(x_3) = -a\left[2\beta_2^2 e^{-\beta_2 x_3}(-B_2\cos\beta_2 x_3 + A_2\sin\beta_2 x_3) + q\right] \quad (29d)$$

where A_1 and B_1 and A_2 and B_2 are the constants of integration, and the coordinate axes x_2 and x_3 are defined separately for each region, as shown in Fig. 3a. Four constants of integration are calculated by imposing transition conditions, given in Eqs. (23a–23d) along contour 2–2. The transition conditions yield the following system of equations to compute the constants of integration:

$$A_1 - A_2 = -\Psi_p^{(\Delta t)} \quad (30a)$$

$$\beta_1(B_1 - A_1) + \beta_2(B_2 - A_2) = 0 \quad (30b)$$

$$-2\frac{\beta_1^2}{E_1h_1}B_1+2\frac{\beta_2^2}{E_2h_2}B_2=\frac{q}{E_2h_2}-\frac{q}{E_1h_1} \tag{30c}$$

$$\frac{\beta_1^3}{E_1h_1}(A_1+B_1)+\frac{\beta_2^3}{E_2h_2}(A_2+B_2)=0 \tag{30d}$$

The solution of Eqs. (30a–30d) yields the four integration constants as

$$A_1=1/\bar{D}\left[\Psi_p^{(\Delta t)}\beta_2^2\left(-2k\beta_1\beta_2-k\beta_1^2-k^2\beta_2^2\right)+\frac{q}{2}(1-k)\left(k\beta_2^2-\beta_1^2\right)\right] \tag{31a}$$

$$A_2=1/\bar{D}\left[\Psi_p^{(\Delta t)}\beta_1^2\left(\beta_1^2+k\beta_2^2+2k\beta_1\beta_2\right)+\frac{q}{2}(1-k)\left(k\beta_2^2-\beta_1^2\right)\right] \tag{31b}$$

$$B_1=1/\bar{D}\left[\Psi_p^{(\Delta t)}k\beta_2^2\left(\beta_1^2-k\beta_2^2\right)+\frac{q}{2\beta_1}(1-k)\left(2k\beta_2^3+k\beta_1\beta_2^2+\beta_1^3\right)\right] \tag{31c}$$

$$B_2=1/\bar{D}\left[\Psi_p^{(\Delta t)}\beta_1^2\left(\beta_1^2-k\beta_2^2\right)-\frac{q}{2\beta_2}(1-k)\left(2\beta_1^3+\beta_1^2\beta_2+k\beta_2^3\right)\right] \tag{31d}$$

where $\bar{D}=2k\beta_1\beta_2(\beta_1^2+\beta_2^2)+(\beta_1^2+k\beta_2^2)^2$ and $k=(E_1h_1)/(E_2h_2)$. The integration constants given in Eqs. (31a–31d) are introduced into Eqs. (29a–29d) to obtain expressions the moment M_x and the force N_φ for both regions I and II. The displacement can then be calculated using the expression $w=aN_\varphi/(Eh)$.

Response near Simply Supported Boundary: Contour 3–3

For this case a procedure similar to that presented for the contour 1–1 is followed. The coordinate axis x_4 is defined as shown in Fig. 3a. Contour 3–3 is simply supported, and the conditions given in Eqs. (21a) and (21b) are applied to obtain the expressions for the internal forces as

$$M_x(x_4)=\left(q/2\beta_2^2\right)e^{-\beta_2x_4}\sin\beta_2x_4 \tag{32a}$$

$$N_\varphi(x_4)=qa\left(e^{-\beta_2x_4}\cos\beta_2x_4-1\right) \tag{32b}$$

and the displacement w is calculated as

$$w(x_4)=\left(qa^2/Eh_2\right)\left(1-e^{-\beta_2x_4}\cos\beta_2x_4\right) \tag{33}$$

Analysis of the composite shell with simply supported, clamped, and interface boundaries can be performed using the CBMF. The problem, however, cannot be solved using classical BMF because the following boundary compatibility conditions were missing: Eqs. (22a) and (22b) for the clamped edge, Eq. (21b) for the simply supported boundary, and Eqs. (23c) and (23d) for the interface boundary.

The solution for force N_φ for the shell with material parameters the same as for the composite plate, $a=4.0$ in., $h_1=0.15$ in., $h_2=0.1$ in., $q=10$ lb/in.^{2} and $\Delta t=10°$F, is depicted in Fig. 3b and discontinuity is noted at the interface. The finite element method will struggle for this case and, at best, can provide an acceptable solution with a dense mesh. In other words, the CBMF represents an elegant approach for this composite shell problem.

Conclusions

The CBMF has been established for the analysis of boundary value problems in elasticity with stress, displacement, and mixed boundary conditions. It is obtained by augmenting the classical Beltrami–Michell formulation with the novel boundary compatibility conditions. The CBMF alleviates the limitations of the classical formulation, which was applicable only for the analysis stress-boundary value problems. All equations of the CBMF have been derived from the stationary condition of the IFM variational functional and specialized for the analysis of circular plates and cylindrical shells subjected to both mechanical and thermal loadings. Transition conditions on interfaces have been established for composite plates and shells. Mixed boundary value problems have been solved using the CBMF to demonstrate its capability to solve mixed boundary value problems. The CBMF may be regarded as an alternate approach to Navier's displacement formulation. It provides theoretical background for the discrete integrated force method which can be used to solve difficult problems with regions of high-stress concentrations.

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